Electrical and Electronics Engineering
2024-2025
Master Semester 2

Course
Smart grids technologies

The branch-flow model and load flow approximations

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Outline

The branch-flow model

The Ward – Hale approximation

The Carpentier approximation

The Stott Approximation

The DC Approximation

The linearized load-flow model

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The branch-flow model

The Ward – Hale approximation

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The DC Approximation

The linearized load-flow model

Recall – The nodal LF model

Let consider **a grid composed by s nodes**. The classical load flow model relies on the following set of **nodal equations**:

$[\bar{\mathbf{I}}] = [\bar{\mathbf{Y}}][\bar{\mathbf{V}}]$ Kirchoff's laws (KVL and KCL) applied to the whole grid;

$\bar{S}_j = \bar{V}_j \underline{I}_j$ Nodal power injection $\forall j = 1, \dots, s$;

$\sum_{j=1}^{s} \bar{S}_j = 0$ Power balance over the whole grid.

Recall the load flow equations in polar coordinates

$$\begin{cases} P_i = \sum_{\ell=1}^{s} V_i V_\ell Y_{i\ell} \cos\left(\vartheta_i - \vartheta_\ell - \gamma_{i\ell}\right) \\ Q_i = \sum_{\ell=1}^{s} V_i V_\ell Y_{i\ell} \sin\left(\vartheta_i - \vartheta_\ell - \gamma_{i\ell}\right) \end{cases}$$

$i=1,2,\dots,g+u$, for the g generator buses + u load buses

$i=g+1,\dots,g+u$ for the load buses

Recall – The nodal LF model

Let consider **a grid composed by s nodes**. The classical load flow model relies on the following set of **nodal equations**:

$[\bar{\mathbf{I}}] = [\bar{\mathbf{Y}}][\bar{\mathbf{V}}]$ Kirchoff's laws (KVL and KCL) applied to the whole grid;

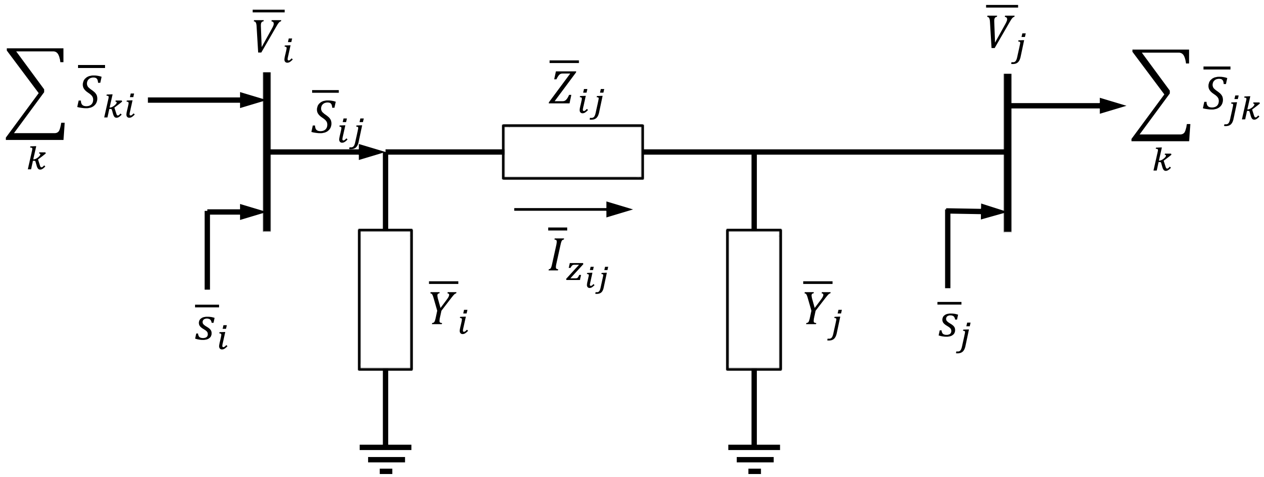
$\bar{S}_j = \bar{V}_j \underline{I}_j$ Nodal power injection $\forall j = 1, \dots, s$;

$\sum_{j=1}^{s} \bar{S}_j = 0$ Power balance over the whole grid.

Type of bus	Independent variable $(2s)$		Dependent variable $(2s)$	
Generator buses	P	$\lvert\bar{V}\rvert$	Q	$arg(\bar{V})$
Load buses	P	Q	$\lvert\bar{V}\rvert$	$arg(\bar{V})$
Slack bus	$\lvert\bar{V}\rvert$	$arg(\bar{V})$	P	Q

The branch flow model

Let consider a **generic branch between nodes i and j of the network modelled by a generic Π-equivalent model**



$\overline{V}_i, \overline{V}_j$: complex nodal voltages respectively at nodes i and j

$\overline{s}_i, \overline{s}_j$: complex apparent power injections respectively at nodes i and j

$\overline{S}_{ij}$: complex power flow from node i to node j

$\overline{I}_{z_{ij}}$: complex current flow through the branch impedance $\overline{Z}_{ij}$

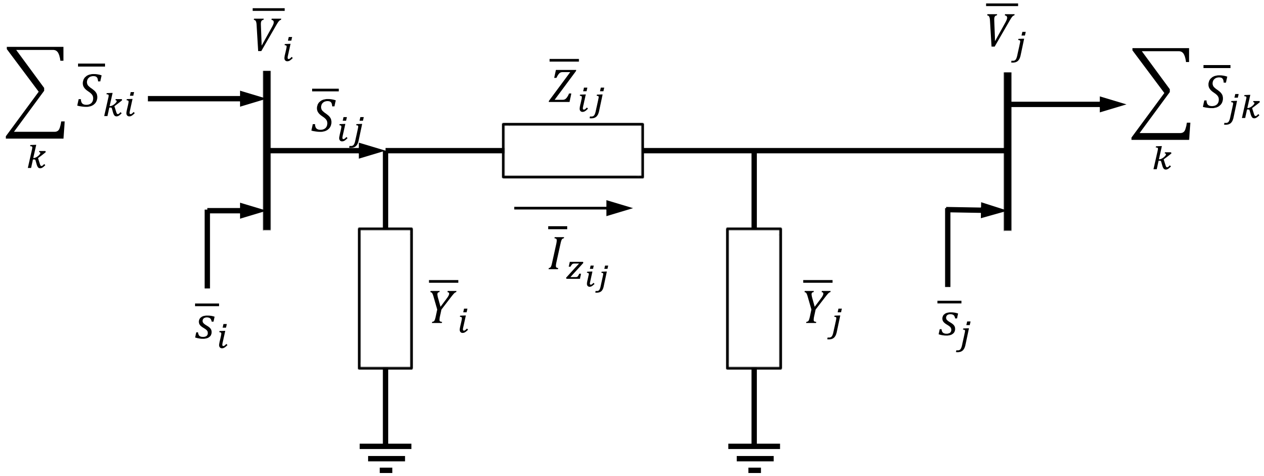
$\overline{Y}_i, \overline{Y}_j$: complex shunt admittances respectively at nodes i and j

$\sum_k \overline{S}_{ki}$: complex apparent power entering into node i from the rest of the grid

$\sum_k \overline{S}_{jk}$: complex apparent power leaving node j towards the rest of the grid

The branch flow model

For the generic branch in the figure, let now write **the Kirchoff's voltage law** and **the power balances**.



$$\bar{V}_i - \bar{V}_j = \bar{Z}_{ij}\bar{I}_{z_{ij}}$$ Kirchoff's voltage law on the branch

$$\sum_k \bar{S}_{ki} + \bar{s}_i = \bar{S}_{ij}$$ Power balance at node i

$$\bar{S}_{ij} - \bar{Z}_{ij}\left|\bar{I}_{z_{ij}}\right|^2 - \bar{Y}_i\left|\bar{V}_i\right|^2 - \bar{Y}_j\left|\bar{V}_j\right|^2 + \bar{s}_j = \sum_k \bar{S}_{jk}$$ Power balance over the branch ij

Observation: by fixing the nodal injections and/or nodal voltages, the set of all grid branches' equations results in a different, but equivalent, load flow model.

The branch flow model cont'd

The previous set of equations **contains quadratic terms of nodal voltages** V_i^2, V_j^2 **and currents** $I_{z_{ij}}^2$. However, it can be further simplified to obtain a set of constraints that is largely used in the context of the **optimal power flow problem.**

Let's **introduce two new variables composed by the squares of magnitudes of nodal voltages and currents**:

$$v_i = \left|\overline{V}_i\right|^2, v_j = \left|\overline{V}_j\right|^2$$

$$i_{z_{ij}} = \left|\overline{I}_{z_{ij}}\right|^2$$

With these new variables, we can **rewrite the power balance equation over the branch** ij **as a linear equation over the new variables**:

$$\overline{S}_{ij} - \overline{Z}_{ij}\left|\overline{I}_{z_{ij}}\right|^2 - \overline{Y}_i\left|\overline{V}_i\right|^2 - \overline{Y}_j\left|\overline{V}_j\right|^2 + \overline{s}_j = \sum_k \overline{S}_{jk}$$

$$\overline{S}_{ij} - \overline{Z}_{ij} i_{z_{ij}} - \overline{Y}_i v_i - \overline{Y}_j v_j + \overline{s}_j = \sum_k \overline{S}_{jk}$$

The branch flow model cont'd

We can also rewrite the Kirchoff's voltage law on the branch

$$\overline{V}_i - \overline{V}_j = \overline{Z}_{ij}\overline{I}_{z_{ij}}$$

$$\overline{V}_j = \overline{V}_i - \overline{Z}_{ij}\overline{I}_{z_{ij}}$$

Let's multiply both sides of the last equation by their complex conjugate, we get:

$$\overline{V}_j \underline{V_j} = \left(\overline{V}_i - \overline{Z}_{ij}\overline{I}_{z_{ij}}\right)\left(\underline{V_i} - \underline{Z}_{ij}\underline{I}_{z_{ij}}\right)$$

$$\overline{V}_j \underline{V_j} = \overline{V}_i \underline{V_i} + \overline{Z}_{ij}\underline{Z}_{ij}\overline{I}_{z_{ij}}\underline{I}_{z_{ij}} - \overline{V}_i\underline{Z}_{ij}\underline{I}_{z_{ij}} - \underline{V_i}\overline{Z}_{ij}\overline{I}_{z_{ij}}$$

In view of the newly introduced variables, we can rewrite the last equation as:

$$v_j = v_i + \left|\overline{Z}_{ij}\right|^2 i_{z_{ij}} - \overline{V}_i\underline{Z}_{ij}\underline{I}_{z_{ij}} - \underline{V_i}\overline{Z}_{ij}\overline{I}_{z_{ij}}$$

The branch flow model cont'd

It is worth noting that $\overline{V}_i \underline{Z}_{ij} \underline{I}_{z_{ij}}$ and $\underline{V}_i \overline{Z}_{ij} \overline{I}_{z_{ij}}$ are **complex conjugate of the same term**. Therefore, **their sum can be written as** $2\Re\left(\underline{Z}_{ij}\overline{V}_i \underline{I}_{z_{ij}}\right)$ and the term $\overline{V}_i \underline{I}_{z_{ij}}$ can be written in terms of powers as

$$\overline{V}_i \underline{I}_{z_{ij}} = \overline{S}_{ij} - \overline{Y}_i \left|\overline{V}_i\right|^2$$

Therefore, we can rewrite the Kirchoff's voltage law on the branch as

$$v_j = v_i + \left|\overline{Z}_{ij}\right|^2 i_{z_{ij}} - 2\Re\left[\underline{Z}_{ij}\left(\overline{S}_{ij} - \overline{Y}_i v_i\right)\right]$$

$$v_j = v_i + \left(R_{ij}^2 + X_{ij}^2\right) i_{z_{ij}} - 2\Re\left[\underline{Z}_{ij}\left(\overline{S}_{ij} - \overline{Y}_i v_i\right)\right]$$

where R_{ij} and X_{ij} are the branch resistance and reactance.

It is interesting to note that this equation is linear with respect to the newly introduced variables v_j, v_i and $i_{z_{ij}}$.

The branch flow model cont'd

Therefore, we have

$$\bar{S}_{ij} - \bar{Z}_{ij} i_{z_{ij}} - \bar{Y}_i v_i - \bar{Y}_j v_j + \bar{s}_j = \sum_k \bar{S}_{jk}$$

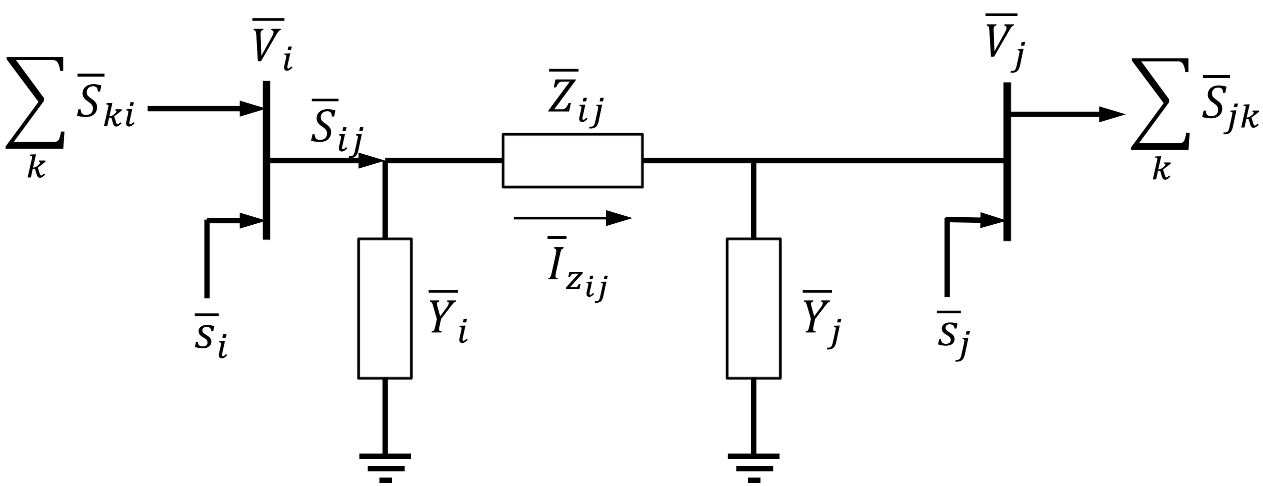
$$v_j = v_i + \left|\bar{Z}_{ij}\right|^2 i_{z_{ij}} - 2\Re\left[\underline{Z}_{ij}\left(\bar{S}_{ij} - \bar{Y}_i v_i\right)\right]$$

$$\sum_k \bar{S}_{ki} + \bar{s}_i = \bar{S}_{ij}$$

where

$v_i = \left|\bar{V}_i\right|^2, v_j = \left|\bar{V}_j\right|^2$

$i_{z_{ij}} = \left|\bar{I}_{z_{ij}}\right|^2$

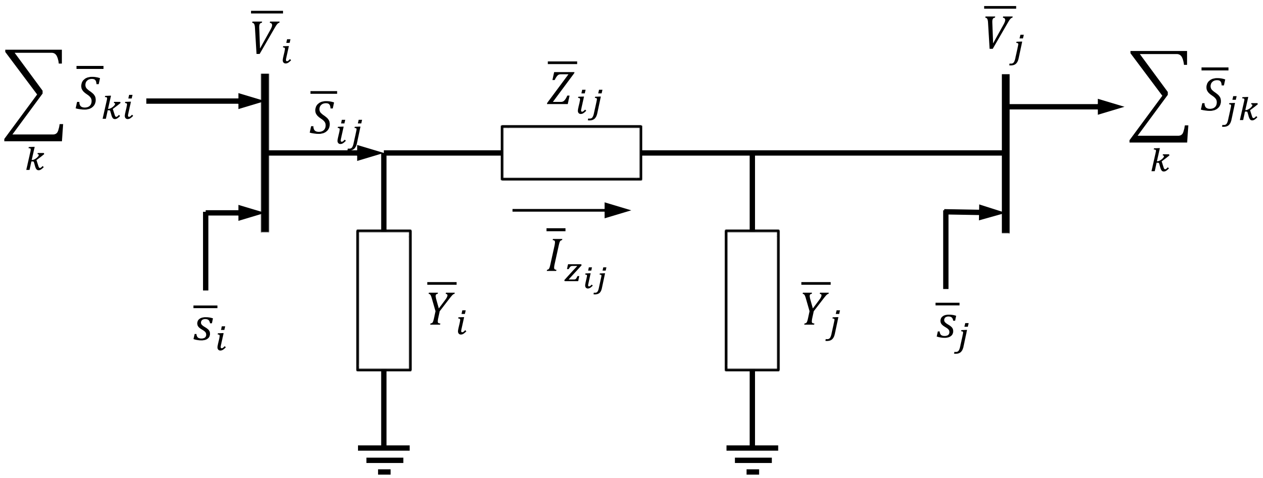


The branch flow model cont'd

It is worth noting that, in the previous set of equations, **we may want to express the current $i_{z_{ij}}$ in terms of voltages and powers.** Therefore, we have that

$$i_{z_{ij}} = \frac{\left|\bar{S}_{ij} - \bar{Y}_i v_i\right|^2}{v_i}$$

This last equation may complete the branch flow model to have all expressed in terms of voltages and powers.



The branch flow model cont'd

In summary, the branch flow model of the load flow equations is given buy the following set of equations:

$$\overline{S}_{ij} - \overline{Z}_{ij} i_{z_{ij}} - \overline{Y}_i v_i - \overline{Y}_j v_j + \overline{s}_j = \sum_k \overline{S}_{jk}$$

$$v_j = v_i + \left|\overline{Z}_{ij}\right|^2 i_{z_{ij}} - 2\Re\left[\underline{Z}_{ij}\left(\overline{S}_{ij} - \overline{Y}_i v_i\right)\right]$$

$$\sum_k \overline{S}_{ki} + \overline{s}_i = \overline{S}_{ij}$$

$$i_{z_{ij}} = \frac{\left|\overline{S}_{ij} - \overline{Y}_i v_i\right|^2}{v_i}$$

Observation: since we have introduced the new variables $v_i = \left|\overline{V}_i\right|^2, v_j = \left|\overline{V}_j\right|^2$ and $i_{z_{ij}} = \left|\overline{I}_{z_{ij}}\right|^2$, we have dropped the current and voltage phases. Therefore, there is the need to recover the voltage phases.

The branch flow model cont'd

To recover the phase of nodal voltages, we may use re-use the Kirchoff's voltage law applied to the branch and its conjugate

$$\overline{V}_j = \overline{V}_i - \overline{Z}_{ij}\overline{I}_{z_{ij}}$$

$$\underline{V_j} = \underline{V_i} - \underline{Z}_{ij}\underline{I}_{z_{ij}}$$

Let's multiply both terms of this last equation by $\overline{V}_i$:

$$\overline{V}_i\underline{V_j} = \overline{V}_i\left(\underline{V_i} - \underline{Z}_{ij}\underline{I}_{z_{ij}}\right)$$

$$\overline{V}_i\underline{V_j} = v_i - \underline{Z}_{ij}\overline{V}_i\underline{I}_{z_{ij}}$$

By recalling that $\overline{V}_i\underline{I}_{z_{ij}} = \overline{S}_{ij} - \overline{Y}_i\left|\overline{V}_i\right|^2$, we have

$$\overline{V}_i\underline{V_j} = v_i - \underline{Z}_{ij}\left(\overline{S}_{ij} - \overline{Y}_i\left|\overline{V}_i\right|^2\right)$$

Therefore, we can derive the equation to recover the phases of nodal voltage phasors:

$$arg\left(\overline{V}_i\right) + arg\left(\underline{V_j}\right) = arg\left(v_i - \underline{Z}_{ij}\left(\overline{S}_{ij} - \overline{Y}_i\left|\overline{V}_i\right|^2\right)\right)$$

Outline

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The Ward – Hale approximation

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The Ward – Hale approximation

Cartesian coordinates

The **Ward - Hale approximation** is related to the application of the Newton-Raphson method to the Load-Flow problem solution using **Cartesian coordinates**.

- According to this approximation, **we consider the variations $[\Delta P]$ and $[\Delta Q]$ of the powers injected into the network in a generic node depending only on the voltage of that node.**

- This approach is justified by the fact that the partial derivatives of the powers injected into the node i with respect to V_i' and V_i'' expressed by (LF.32) and (LF.33) are, in general, sufficiently larger than V_l' and V_l'' for $i \neq l$.

$$J_{PR}:\begin{cases} \dfrac{\partial P_i}{\partial V_\ell'} = G_{i\ell} V_i' + B_{i\ell} V_i'' \\ \dfrac{\partial P_i}{\partial V_i'} = 2G_{ii} V_i' + \sum_{\substack{\ell=1 \\ \ell \neq i}}^{s} \left(G_{i\ell} V_\ell' - B_{i\ell} V_\ell'' \right) \end{cases} \qquad (\text{LF.32})$$

$$J_{PX}:\begin{cases} \dfrac{\partial P_i}{\partial V_\ell''} = -B_{i\ell} V_i' + G_{i\ell} V_i'' \\ \dfrac{\partial P_i}{\partial V_i''} = 2G_{ii} V_i'' + \sum_{\substack{\ell=1 \\ \ell \neq i}}^{s} \left(B_{i\ell} V_\ell' + G_{i\ell} V_\ell'' \right) \end{cases} \qquad (\text{LF.33})$$

The Ward – Hale approximation

Cartesian coordinates

- $\frac{\partial P_i}{\partial V_i'}$ and $\frac{\partial P_i}{\partial V_i''}$ contain a summation term, tacking into consideration all the connections afferent to the node i
- $\frac{\partial P_i}{\partial V_l'}$ and $\frac{\partial P_i}{\partial V_l''}$ contain a term for the single connections between the node in examination and the other node of the grid to which the derivative refers (l).

$$J_{PR}: \begin{cases} \dfrac{\partial P_i}{\partial V_\ell'} = G_{i\ell} V_i' + B_{i\ell} V_i'' \\ \dfrac{\partial P_i}{\partial V_i'} = 2G_{ii} V_i' + \sum_{\substack{\ell=1 \\ \ell \neq i}}^{s} \left(G_{i\ell} V_\ell' - B_{i\ell} V_\ell'' \right) \end{cases} \quad \text{(LF.32)}$$

$$J_{PX}: \begin{cases} \dfrac{\partial P_i}{\partial V_\ell''} = -B_{i\ell} V_i' + G_{i\ell} V_i'' \\ \dfrac{\partial P_i}{\partial V_i''} = 2G_{ii} V_i'' + \sum_{\substack{\ell=1 \\ \ell \neq i}}^{s} \left(B_{i\ell} V_\ell' + G_{i\ell} V_\ell'' \right) \end{cases} \quad \text{(LF.33)}$$

- It is therefore possible to assume:

$$\frac{\partial P_i}{\partial V_l'} = \frac{\partial P_i}{\partial V_l''} \approx 0 \qquad\qquad \frac{\partial Q_i}{\partial V_l'} = \frac{\partial Q_i}{\partial V_l''} \approx 0$$

The Ward – Hale approximation

Cartesian coordinates

As a direct consequence, the approximation consists in assuming the matrix $[Y]$ of nodal admittances to be a **diagonally dominant matrix.**

According to the Ward-hale approximation, the Jacobian submatrices:

$$\left[\frac{\partial P}{\partial V'}\right] \quad \left[\frac{\partial P}{\partial V''}\right] \quad \left[\frac{\partial Q}{\partial V'}\right] \quad \left[\frac{\partial Q}{\partial V''}\right]$$

become purely diagonal, as they present **non-zero elements only on the diagonal**. Since the matrices:

$$\left[\frac{\partial (V^2)}{\partial V'}\right] \quad \left[\frac{\partial (V^2)}{\partial V''}\right]$$

are diagonal as well , it turns out that **all six Jacobian sub-matrices are diagonal, resulting in a decoupling of the iteration process.**

The Ward – Hale approximation

Cartesian coordinates

With the Ward-Hale approximation, the equation (LF.27) for **load buses** becomes:

(LF.27)

$$\begin{bmatrix} \dfrac{\partial P}{\partial V'} & \dfrac{\partial P}{\partial V''} \\ \dfrac{\partial Q}{\partial V'} & \dfrac{\partial Q}{\partial V''} \end{bmatrix}^{(\nu)} \times \begin{bmatrix} \Delta V' \\ \Delta V'' \end{bmatrix}^{(\nu+1)} = \begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix}^{(\nu)} \Rightarrow \begin{cases} \Delta P_i^{(\nu)} = \dfrac{\partial P_i^{(\nu)}}{\partial V_i'} \Delta V_i'^{(\nu+1)} + \dfrac{\partial P_i^{(\nu)}}{\partial V_i''} \Delta V_i''^{(\nu+1)} \\ \Delta Q_i^{(\nu)} = \dfrac{\partial Q_i^{(\nu)}}{\partial V_i'} \Delta V_i'^{(\nu+1)} + \dfrac{\partial Q_i^{(\nu)}}{\partial V_i''} \Delta V_i''^{(\nu+1)} \end{cases}$$

While for the **generator buses** (LF.36):

(LF.36)

$$\begin{bmatrix} \dfrac{\partial P}{\partial V'} & \dfrac{\partial P}{\partial V''} \\ \dfrac{\partial V^2}{\partial V'} & \dfrac{\partial V^2}{\partial V''} \end{bmatrix}^{(\nu)} \times \begin{bmatrix} \Delta V' \\ \Delta V'' \end{bmatrix}^{(\nu+1)} = \begin{bmatrix} \Delta P \\ \Delta V^2 \end{bmatrix}^{(\nu)} \Rightarrow \begin{cases} \Delta P_i^{(\nu)} = \dfrac{\partial P_i^{(\nu)}}{\partial V_i'} \Delta V_i'^{(\nu+1)} + \dfrac{\partial P_i^{(\nu)}}{\partial V_i''} \Delta V_i''^{(\nu+1)} \\ \Delta \left(V_i^2\right)^{(\nu)} = \dfrac{\partial \left(V_i^2\right)^{(\nu)}}{\partial V_i'} \Delta V_i'^{(\nu+1)} + \dfrac{\partial \left(V_i^2\right)^{(\nu)}}{\partial V_i''} \Delta V_i''^{(\nu+1)} \end{cases}$$

Both systems have 2 equations with the 2 unknown $\Delta V_i'$ and $\Delta V_i''$. Passing to the next node $(i+1)$ in the same iteration, it is possible to use the updated value of voltage V_i', V_i'' in the computation of:

$$\left[\frac{\partial P_i}{\partial V_i'}\right] \quad \left[\frac{\partial P_i}{\partial V_i''}\right] \quad \left[\frac{\partial Q_i}{\partial V_i'}\right] \quad \left[\frac{\partial Q_i}{\partial V_i''}\right]$$

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Recall

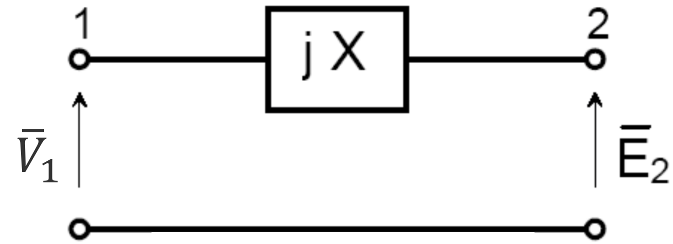
Electrically short lines

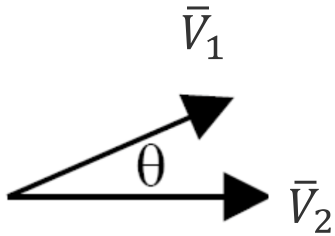
Let us consider a very short line, that is, a line in which the transverse admittance can be neglected:

$$P_1 = P_2 = \frac{3V_1V_2}{X}\sin\theta$$

$$Q_1 = \frac{3V_1}{X}(V_1 - V_2\cos\theta)$$

$$Q_2 = \frac{3V_2}{X}(V_1\cos\theta - V_2)$$





- The **active power** depends on **the angle difference between V_1 and V_2**.
- Being θ a generally very small angle, it can be affirmed that $\cos\theta \approx 1$ and therefore the **reactive power** depends mainly on the **modules of the voltages**.

The Carpentier approximation

Polar coordinates

The **Carpentier method** consists in supposing that the **active powers** injected into the nodes depend only **on the phases of the voltages** and that the **reactive powers** depend only on the **modules of the voltages (i.e. active-reactive decoupling).**

- Decoupling between the active power variables (i.e. voltage phases) and reactive powers (i.e. voltage modules).
- The elements of the sub-matrix $\left[\frac{\partial P}{\partial \theta}\right]$ are, **for high voltage transmission systems,** normally much larger than those of the sub-matrix $\left[\frac{\partial P}{\partial V}\right]$. **Note that this is not true in distribution systems.**
- The elements of the sub-matrix $\left[\frac{\partial Q}{\partial V}\right]$ are, **for high voltage transmission systems,** normally much larger than those of the sub-matrix $\left[\frac{\partial Q}{\partial \theta}\right]$. **Note that this is not true in distribution systems.**

- Therefore:

$$\left[\frac{\partial P}{\partial \theta}\right] \gg \left[\frac{\partial P}{\partial V}\right] \quad \Longrightarrow \quad \left[\frac{\partial P}{\partial V}\right] \approx 0$$

$$\left[\frac{\partial Q}{\partial V}\right] \gg \left[\frac{\partial Q}{\partial \theta}\right] \quad \Longrightarrow \quad \left[\frac{\partial Q}{\partial \theta}\right] \approx 0$$

The Carpentier approximation

Polar coordinates

- The approximation of Carpentier consists in supposing null the matrices $\left[\frac{\partial P}{\partial V}\right]$ and $\left[\frac{\partial Q}{\partial \theta}\right]$ for which the system (LF.42) becomes:

$$\begin{bmatrix} \frac{\partial P}{\partial V} & \frac{\partial P}{\partial \vartheta} \\ \frac{\partial Q}{\partial V} & \frac{\partial Q}{\partial \vartheta} \end{bmatrix}^{(\nu)} \times \begin{bmatrix} \Delta V \\ \Delta \vartheta \end{bmatrix}^{(\nu+1)} = \begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix}^{(\nu)} \quad \text{(LF.42)} \qquad \Longrightarrow \begin{bmatrix} 0 & \frac{\partial P}{\partial \theta} \\ \frac{\partial Q}{\partial V} & 0 \end{bmatrix}^{(\nu)} \times \begin{bmatrix} \Delta V \\ \Delta \theta \end{bmatrix}^{(\nu+1)} = \begin{bmatrix} \Delta P \\ \Delta Q \end{bmatrix}^{(\nu)}$$

- Therefore, in two independent equations:

$$\left[\frac{\partial P}{\partial \theta}\right]^{(\nu)} \times [\Delta\theta]^{(\nu+1)} = [\Delta P]^{(\nu)}$$

$$\left[\frac{\partial Q}{\partial V}\right]^{(\nu)} \times [\Delta V]^{(\nu+1)} = [\Delta Q]^{(\nu)}$$

Full uncoupling between active and reactive components.

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Load flow in polar coordinates

Starting from the polar coordinate formulation of Load Flow (slide 14 Lecture 2.3):

$$\bar{S_i} = P_i + jQ_i = \bar{V_i}\underline{I}_i = \bar{V_i}\sum_{l=1}^{S} \underline{Y}_{il}\,\underline{V}_l = \sum_{l=1}^{S} \bar{V_i}\;\underline{Y}_{il}\,\underline{V}_l = \sum_{l=1}^{S} V_i\,V_l Y_{il} e^{j(\theta_i-\theta_l-\gamma_{il})}$$

By imposing:

$$\begin{cases} \bar{V_i} & = V_i e^{j\theta_i} \quad = V_i[\cos\theta_i + j\sin\theta_i] \\ \underline{V_l} & = V_l e^{-j\theta_l} \quad = V_l[\cos(-\theta_l) + j\sin(-\theta_l)] \\ \underline{Y}_{il} & = G_{il} - jB_{il} \end{cases}$$

$$\bar{S}_i = V_i\sum_{l=1}^{S} V_l(G_{il} - jB_{il})[\cos(\theta_i - \theta_l) + j\sin(\theta_i - \theta_l)] = V_i\sum_{l=1}^{S} V_l(G_{il} - jB_{il})[\cos(\theta_{il}) + j\sin(\theta_{il})]$$

The Stott Approximation

Polar coordinates

From which (same as slide 18 Lecture 2.3):

$$\begin{cases} P_i = V_i \sum_{l=1}^{S} V_l (G_{il} \cos\theta_{il} + B_{il} \sin\theta_{il}) \\ Q_i = V_i \sum_{l=1}^{S} V_l (G_{il} \sin\theta_{il} - B_{il} \cos\theta_{il}) \end{cases}$$

extracting from the sum the term relative to the node i

$$\begin{cases} P_i = +G_{ii} V_i^2 + V_i \sum_{\substack{l=1 \\ l \neq i}}^{S} V_l (G_{ih} \cos\theta_{il} + B_{il} \sin\theta_{il}) \\ Q_i = -B_{ii} V_i^2 + V_i \sum_{\substack{l=1 \\ l \neq i}}^{S} V_l (G_{il} \sin\theta_{il} - B_{il} \cos\theta_{il}) \end{cases}$$

The Stott Approximation

Polar coordinates

By calculating the derivatives:

$$\frac{\partial P_i}{\partial \theta_i} = -V_i \sum_{l \neq i} V_l (G_{il} \sin \theta_{il} - B_{il} \cos \theta_{il})$$

$$\frac{\partial Q_i}{\partial V_i} = -2B_{ii}V_i + \sum_{l \neq i} V_l (G_{il} \sin \theta_{il} - B_{il} \cos \theta_{il})$$

Hypothesis:

1. $B_{il}\, cos\, \theta_{il} \approx B_{il}$ since the angle differences θ_{il} are small, i.e. $cos\, \theta_{il} \approx 1$
2. $G_{il}\, sin\, \theta_{il} \ll B_{il}$ since the values of G_{il} are extremely small
3. $Q_i \ll B_{ii} V_i^2$

The Stott Approximation

Polar coordinates

As a consequence:

$$\begin{cases} \dfrac{\partial P_i}{\partial \theta_l} = -V_i V_l B_{ih} \\ \dfrac{\partial P_i}{\partial \theta_i} = -V_i^2 B_{ii} \end{cases} \quad l \neq i \qquad \Longrightarrow \qquad \frac{\partial P_i}{\partial \theta_l} = -V_i V_l B_{il} \quad \forall l$$

$$\begin{cases} \dfrac{\partial Q_i}{\partial V_l} = -V_i B_{il} \\ \dfrac{\partial Q_i}{\partial V_i} = -V_i B_{ii} \end{cases} \quad l \neq i \qquad \Longrightarrow \qquad \frac{\partial Q_i}{\partial V_l} = -V_i B_{il} \quad \forall l$$

As a consequence, the equation for the LF:

$$\begin{cases} \Delta P_i^{(\nu)} = \displaystyle\sum_{l=1}^{S} \frac{\partial P_i^{(\nu)}}{\partial \theta_l} \Delta \theta_l^{(\nu+1)} \\ \Delta Q_i^{(\nu)} = \displaystyle\sum_{l=1}^{S} \frac{\partial Q_i^{(\nu)}}{\partial V_l} \Delta V_l^{(\nu+1)} \end{cases} \Longrightarrow \begin{cases} \dfrac{\Delta P^{(\nu)}}{V_i} = \displaystyle\sum_{l=1}^{S} (-B_{i,l}) V_i^{(\nu+1)} \Delta \theta_l^{(\nu+1)} \\ \dfrac{\Delta Q^{(\nu)}}{V_i} = \displaystyle\sum_{l=1}^{S} (-B_{i,l}) \Delta V_l^{(\nu+1)} \end{cases}$$

Jacobian matrices have become (constant) B matrices.
Therefore, the NR method does not require any iteration.

Summary

Ward-Hale

Coordinates:
Cartesian

Hypothesis:
$[\Delta P]$ and $[\Delta Q]$ in a generic node are depending only on the voltage of that node

$$\frac{\partial P_i}{\partial V_l'} = \frac{\partial P_i}{\partial V_l''} = 0 \quad i \neq l$$

$$\frac{\partial Q_i}{\partial V_l'} = \frac{\partial Q_i}{\partial V_l''} = 0 \quad i \neq l$$

Result:
Decoupling between nodes

Carpentier

Coordinates:
Polar

Hypothesis:
P is only function of θ and Q of V

$$\left[\frac{\partial P}{\partial V}\right] \approx 0$$

$$\left[\frac{\partial Q}{\partial \theta}\right] \approx 0$$

Result:
Decoupling of P and Q

Stott

Coordinates:
Polar + Cartesian (mixed)

Hypothesis:
P is only function of θ and Q is function of V

$$\left[\frac{\partial P}{\partial V}\right] \approx 0 \qquad \left[\frac{\partial Q}{\partial \theta}\right] \approx 0$$

$$B_{il}\, cos\, \theta_{il} \approx B_{il}$$

$$G_{il}\, sin\, \theta_{il} \ll B_{il}$$

$$Q_i \ll B_{ii} V_i^2$$

Result:
Decoupling of P and Q
No iterations

Outline

The branch-flow model

The Ward – Hale approximation

The Carpentier approximation

The Stott Approximation

The DC Approximation

The linearized load-flow model

The DC Approximation

- **For high voltage systems, the longitudinal resistances of the line conductors and copper losses of transformers are neglected with respect to the series reactance of the lines and transformers (note that this is not true in distribution systems).**
  - Acceptable when the calculation of losses is waived.
  - $\frac{x}{r} \approx 10$ for transmission lines,
  - $\frac{x}{r} \approx 50$ for transformers
  - In both cases $\bar{z} \approx jx$
- **The transverse admittances of the network components are neglected.**
  - The shunt capacitances of the lines generate reactive power especially in long lines at very high voltage;
  - The currents flowing through the shunt capacitances are mainly associated to the reactive power balance of the line and, in high voltage systems, they are related to the difference of voltages magnitudes at the extremes of the lines.
  - In high voltage systems, however, shunt capacitances have a little influence on the active power flows that mainly depend on the differences between the phases of the voltage phasors at the line ends.
  - The shunt conductance, which take into account the corona and insulators losses of the lines and the iron-losses transformers, may assumed small and, in the DC approximation, negligible.

With these simplifications, **the grid model is only composed by longitudinal inductive reactances** (i.e., the equivalent series reactance of lines and transformers).

The DC Approximation

- $\bar{V}_i$, $\bar{V}_l$ are the voltages phasors at the extremities
- θ_i is the argument of $\bar{V}_i$ and θ_l the argument of $\bar{V}_l$, $\theta_{il} = \theta_i - \theta_l$
- x_{il} the reactance of the branch il

In view of the above, the active power through the branch il is:

$$P_{il} = \frac{3V_iV_l}{x_{il}} \sin\theta_{il}$$

Further hypotheses of the DC approximation are the following:

- the modules of the nodal voltages all equal to $1\ pu$
- the difference $\theta_i - \theta_l$ is small, therefore $sin(\theta_i - \theta_l) \approx \theta_i - \theta_l$

Therefore, we have

$$P_{il} = \frac{1}{x_{il}}\theta_{il}$$

As a consequence, the injection of power in a generical node i is :

$$P_i = \sum_{l \neq i} P_{il} = \frac{\theta_{i1}}{x_{i1}} + \cdots \frac{\theta_{is}}{x_{is}}$$

(note that the voltage angles are known up to one phase shift, so we need to take one node, say k, as reference and set $\theta_k = 0$ at this node), so we have that:

$$P_i = \left(\frac{1}{x_{i1}} + \cdots \frac{1}{x_{is}}\right)\theta_i - \sum_{\substack{l=1 \\ l \neq i}}^{s} \frac{1}{x_{il}}\theta_l = \sum_{l=1}^{s} B_{il}\theta_l$$

The DC Approximation

The following linear matrix equation is obtained for the whole network:

$$[P] = [B] \times [\theta]$$

where $[B]$ and the "susceptance matrix" of the entire transmission network (in pu).

As seen before, the diagonal terms B_{ii} of $[B]$ consist of the sum of the (longitudinal) susceptance of all sides converging at the i-th node

The terms on the diagonal B_{ii} are positive if the susceptance are inductive while the other terms are all negative, provided that the susceptance are inductive, and meet condition

$$B_{lk} = B_{kl}$$

Outline

The branch-flow model

The Ward – Hale approximation

The Carpentier approximation

The Stott Approximation

The DC Approximation

The linearized load-flow model

State-dependent computation of voltage sensitivity coefficients with respect to power injections.

Let consider a generic 3-phase (a,b,c) network with s buses

- $\mathcal{N}$: set of PQ buses
- $\mathcal{H}$: set of slack buses

$$\{1,2,\ldots,s\} = \mathcal{H} \cup \mathcal{N}, \qquad \mathcal{H} \cap \mathcal{N} = \emptyset$$

- **Use of compound $[\bar{\boldsymbol{Y}}_{\boldsymbol{abc}}]$ matrix :**
  - Unbalanced configuration
  - Coupling between phases
  - Highly-sparse

- **STEP 1:**
  - **Nodal equations linking bus voltages to current injections:**
    - $3 \times s$ equations $\qquad [\bar{\boldsymbol{I}}_{\boldsymbol{abc}}] = [\bar{\boldsymbol{Y}}_{\boldsymbol{abc}}] \cdot [\bar{\boldsymbol{V}}_{\boldsymbol{abc}}]$

Efficient Computation of Voltage Sensitivity Coefficients

- **STEP 2:**
  - **The i-th element of the equations can be expressed:**

$$\bar{I}_i = \sum_{j \in \mathcal{H} \cup \mathcal{N}} \bar{Y}_{ij} \bar{V}_j \qquad i \in N$$

- **STEP 3:**
  - **Elimination of currents:**
    - ➢ Link between PQ injections and bus voltages

$$\underline{S}_i = \underline{V}_i \sum_{j \in \mathcal{H} \cup \mathcal{N}} \bar{Y}_{ij} \bar{V}_j \qquad i \in N$$

Sensitivities w.r.t. power injections

➢ Partial derivatives w.r.t. PQ injections

$$\frac{\partial |\bar{V}_i|}{\partial P_l} \qquad \frac{\partial |\bar{V}_i|}{\partial Q_l}, \qquad l \in \mathcal{N}$$

Sensitivities w.r.t. OLTC

➢ Partial derivatives w.r.t. slack bus voltage

$$\frac{\partial |\bar{V}_i|}{\partial \bar{V}_k}, \qquad k \in \mathcal{H}$$

Voltage Sensitivities with respect to power injections

- **STEP 4:**
  - **Partial derivatives w.r.t. PQ injections**

$$\underline{S}_i = \underline{V}_i \sum_{j \in \mathcal{H} \cup \mathcal{N}} \bar{Y}_{ij}\, \bar{V}_j \quad \Longrightarrow \quad \frac{\partial \underline{S}_i}{\partial P_l} = \frac{\partial}{\partial P_l}\left\{\underline{V}_i \sum_{j \in \mathcal{H} \cup \mathcal{N}} \bar{Y}_{ij}\, \bar{V}_j\right\}$$

- **Assumptions:**
  - **Fixed slack bus voltage** $\dfrac{\partial \bar{V}_j}{\partial P_l} = 0, \qquad \forall j \in \mathcal{H}$
  - $\dfrac{\partial \underline{S}_i}{\partial P_l} = \dfrac{\partial \{P_i - jQ_i\}}{\partial P_l} = \mathbf{1}_{\{i=l\}}$ **and** $\dfrac{\partial \underline{S}_i}{\partial Q_l} = \dfrac{\partial \{P_i - jQ_i\}}{\partial Q_l} = -j\mathbf{1}_{\{i=l\}}$

- **Sparse linear system** in rectangular coordinates:

$$\mathbf{1}_{\{i=l\}} = \frac{\partial \underline{V}_i}{\partial P_l} \sum_{j \in \mathcal{H} \cup \mathcal{N}} \bar{Y}_{ij}\, \bar{V}_j + \underline{V}_i \sum_{j \in \mathcal{N}} \bar{Y}_{ij} \frac{\partial \bar{V}_j}{\partial P_l}$$

(*)

$$-j\mathbf{1}_{\{i=l\}} = \frac{\partial \underline{V}_i}{\partial Q_l} \sum_{j \in \mathcal{H} \cup \mathcal{N}} \bar{Y}_{ij}\, \bar{V}_j + \underline{V}_i \sum_{j \in \mathcal{N}} \bar{Y}_{ij} \frac{\partial \bar{V}_j}{\partial Q_l}$$

Voltage Sensitivities with respect to power injections

- **STEP 5:**
  - **Separate in real and imaginary parts**
  - **Solve the linear system for** $\frac{\partial \underline{V}_{i_{REAL}}}{\partial P_l}, \frac{\partial \underline{V}_{i_{IMAG}}}{\partial P_l}$ **and** $\frac{\partial \underline{V}_{i_{REAL}}}{\partial Q_l}, \frac{\partial \underline{V}_{i_{IMAG}}}{\partial Q_l}$
  - **Reconstruct** $\frac{\partial \underline{V}_i}{\partial P_l} = \frac{\partial \underline{V}_{i_{REAL}}}{\partial P_l} + \mathrm{j}\frac{\partial \underline{V}_{i_{IMAG}}}{\partial P_l}$ **and** $\frac{\partial \underline{V}_i}{\partial Q_l} = \frac{\partial \underline{V}_{i_{REAL}}}{\partial Q_l} + \mathrm{j}\frac{\partial \underline{V}_{i_{IMAG}}}{\partial Q_l}$

- **Sensitivity Coefficients $\boldsymbol{K_{P,Q}}$:**

$$K_P^{il} = \frac{\partial |\bar{V}_i|}{\partial P_l} = \frac{1}{|\bar{V}_i|} \, Re\left(\underline{V_i} \, \frac{\partial \, \bar{V}_j}{\partial \, P_l}\right)$$

$$K_Q^{il} = \frac{\partial |\bar{V}_i|}{\partial Q_l} = \frac{1}{|\bar{V}_i|} \, Re\left(\underline{V_i} \, \frac{\partial \, \bar{V}_j}{\partial \, Q_l}\right)$$

The linearized load-flow model

Theorem:
The system of equations (*), which is linear with respect to rectangular coordinates, has a unique solution for every radial electrical network and for any operating point where the load-flow Jacobian is invertible.

- **Proof:**

The system is linear in rectangular coordinates and has as many equations as unknowns. The theorem is equivalent to showing that the homogeneous system of equations has only the trivial solution. The homogeneous system can be written as:

$$0 = \underline{\Delta}_i \sum_{j \in \mathcal{H} \cup \mathcal{N}} \bar{Y}_{ij} \, \bar{V}_j + \underline{V}_i \sum_{j \in \mathcal{N}} \bar{Y}_{ij} \, \bar{\Delta}_j \,, \qquad \forall i \in \mathcal{N}$$

Where $\bar{\Delta}_i$ are the unknown complex numbers. We want to show that they are equal to zero for all $i \in \mathcal{N}$. Let's consider two networks with the same topology (i.e. same $[Y_{abc}]$ matrix), where the voltages are given. In the first network the voltages are:

$$\bar{V}_i' = \bar{V}_i \qquad \forall i \in \mathcal{H}$$
$$\bar{V}_i' = \bar{V}_i + \bar{\Delta}_i \qquad \forall i \in \mathcal{N}$$

In the second the voltages are:

$$\bar{V}_i'' = \bar{V}_i \qquad \forall i \in \mathcal{H}$$
$$\bar{V}_i'' = \bar{V}_i - \bar{\Delta}_i \qquad \forall i \in \mathcal{N}$$

The linearized load-flow model

- **Proof (ctd):**

Let $\underline{S}_i'$ be the conjugate of the absorbed/injected power at the i-th bus in the first network and $\underline{S}_i''$ in the second. Thus, we have the following equations.

Network 1:

$$\underline{S}_i' = \underline{V}'_i \sum_{j\in\mathcal{H}\cup\mathcal{N}} \bar{Y}_{ij}\ \bar{V}_j' = (\underline{V}_i + \underline{\Delta}_i)\left(\sum_{j\in\mathcal{H}} \bar{Y}_{ij}\,\bar{V}_j + \sum_{j\in\mathcal{N}} \bar{Y}_{ij}\ (\bar{V}_j + \bar{\Delta}_j)\right)$$

$$= \underline{V}_i \sum_{j\in\mathcal{H}\cup\mathcal{N}} \bar{Y}_{ij}\ \bar{V}_j + \underline{\Delta}_i \sum_{j\in\mathcal{N}} \bar{Y}_{ij}\,\bar{\Delta}_j + \underline{\Delta}_i \sum_{j\in\mathcal{H}\cup\mathcal{N}} \bar{Y}_{ij}\ \bar{V}_j + \underline{V}_i \sum_{j\in\mathcal{N}} \bar{Y}_{ij}\,\bar{\Delta}_j$$

Network 2:

$$\underline{S}_i'' = \underline{V}_i \sum_{j\in\mathcal{H}\cup\mathcal{N}} \bar{Y}_{ij}\ \bar{V}_j + \underline{\Delta}_i \sum_{j\in\mathcal{N}} \bar{Y}_{ij}\bar{\Delta}_j - \underline{\Delta}_i \sum_{j\in\mathcal{H}\cup\mathcal{N}} \bar{Y}_{ij}\bar{V}_j - \underline{V}_i \sum_{j\in\mathcal{N}} \bar{Y}_{ij}\ \bar{\Delta}_j$$

Subtracting the two we get:

$$\underline{S}_i' - \underline{S}_i'' = 2\left(\underline{\Delta}_i \sum_{j\in\mathcal{H}\cup\mathcal{N}} \bar{Y}_{ij}\bar{V}_j + \underline{V}_i \sum_{j\in\mathcal{N}} \bar{Y}_{ij}\ \bar{\Delta}_j\right)$$

The linearized load-flow model

- **Proof (ctd):**

If we suppose that **the load-flow Jacobian matrix is invertible**, we can **apply the inverse function theorem**. As a consequence, the nonlinear system of the power flow equations is **locally invertible in a neighbourhood around the current operating point**. Now, we take arbitrarily small, such that $\underline{V}_i' = \underline{V}_i''$ belong to this neighbourhood where there is a one-to-one mapping between the powers and voltages. As the powers that correspond to $\underline{V}_i' = \underline{V}_i''$ are exactly the same, it follows that $\underline{S}_i' = \underline{S}_i''$ for every bus $i \in N$. Thus, the two networks have the same active and reactive power at all non-slack busses and the same voltages at all slack busses. Therefore, it follows that the voltage profiles of these networks must be exactly the same, i.e.:

$$\bar{V}_i - \overline{\Delta}_i = \bar{V}_i + \overline{\Delta}_i$$

$$\overline{\Delta}_i = 0$$